On the evaluation of Brewer's character sums

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To show that X, U, V, and W also satisfy XW = V² − U² − UV, solve for ... in terms of X, U, V, W and constant and substitute into the last equation in (9-5).

By setting ... and comparing with (3-1), one may deduce that ...

If p ≢ 1 (mod 10), Φ(1) = −1, by (5.2). If p ≡ 1 (mod 10), ... by (9.5). ... is an immediate consequence.

$\Lambda_{10}(1)$ was given by Brewer [1]. X satisfies the equations in Theorem 8. Its sign is chosen so that X ≡ 4 (mod 5). That X is thus defined uniquely can be deduced from ...

10. Conclusion. The evaluations of Brewer sums presented here may serve ... Since, according to (3.4), ... so that ... means of cyclotomy. There may be further extensions in other directions, as suggested by [3].

This paper is comprised of the principal contributions of the dissertation of the first author, completed under the direction of the second; several results of the latter ... completed under Professor Albert Whiteman.

The formulation of many of the theorems in this paper was aided considerably by studying tables of various character sums. We wish to express our gratitude to the University of Pittsburgh Computer Center for granting access to its IBM 7090/1401 system, partially supported by National Science Foundation grant G-11309.

Theorem 8A. X, U, V and W are integers satisfying ... XW = V² − U² − UV.

Proof. ... is odd, while ... are even [15, p. 123]. From the last equation ... all products of two even numbers ...

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$$\Lambda_{14}(Q) = \begin{cases} 2Y + 7(-1)^f D_{28}(7, 14), & p \equiv 1 \pmod{28}, \\ \ldots & p \equiv 13 \pmod{28}, \\ 0, & \text{otherwise}. \end{cases} \quad (7.3)$$

(9.6) Thus $p \equiv 1 \pmod{14}$, $\Phi_7(1) = 7B_7(-2\ \mathrm{Ind}\ 2, 1) - (p+1)$, by (3.21); otherwise $\Phi_1(1) = -1$, by (5.2). By (7.3):

$$\Lambda_{14}(Q) = \begin{cases} 7B_7(-2\ \mathrm{Ind}\ 2, 1) - (p-1) + \chi(Q)[X + 7(-1)^f D_{28}(0, 14)] \\ -1 + \chi(Q)[-X - 7(-1)^f d_{28}] \\ 7B_7(-2\ \mathrm{Ind}\ 2, 1) - (p-1) \\ -1, \quad \text{otherwise} \end{cases}$$

$B_7(-2\ \mathrm{Ind}\ 2, 1)$ is embedded in a six-variable quadratic decomposition of $72p$ [8, (33)].

9. Another expression for $\Lambda_{10}(Q)$. The Jacobi sum $R(1, 1) = B_3(0, 1) + 2B_3(1, 1) + B_3(2, 1)$ is associated with the binary quadratic form $4p = L^2 + 27M^2$, $L \equiv 1 \pmod 3$:

(9.2) $L = 2B_3(0, 1) - B_3(1, 1) - B_3(2, 1)$, $3M = B_3(1, 1) - B_3(2, 1)$. [7, p. 397]

$R_5(1, 1)$ is associated with the quaternary quadratic form $16p = x^2 + 50u^2 + 50v^2 + 125w^2$, $xw = v^2 - u^2 - 4uv$, $x \equiv 1 \pmod 5$. [7, p. 402]

In studying Jacobi sums of order six one encounters a second binary quadratic form for primes $p \equiv 1 \pmod 3$:

(9.4) $p = A^2 + 3B^2$, $A \equiv 1 \pmod 3$. [7, pp. 408-410]

Relating this form, already generalized in [?], to some theorems of Brewer, the index of 2, modulo 3. However, in (9.4), p has a coefficient of 1. This form, furthermore, is used in evaluating $\Lambda_6(Q)$, $p \equiv 1 \pmod 3$. Notice that in (5.19) there is no reference to Ind 2.

By contrast, the evaluation of $\Lambda_{10}(Q)$ in (7.?) which is expressed in terms of the coordinates of (9.3) has five cases, depending upon Ind 2 (mod 5). One may ask for another quaternary quadratic form for $p \equiv 1 \pmod 5$ which would provide a simpler expression for $\Lambda_{10}(Q)$. Such a form is developed here.

(Attempts to generalize to $p \equiv 1 \pmod 7$ failed; an appropriate quadratic form in six variables was not discovered.)

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$$A_{10}(g) = \chi(2)A_5(1) + 1 + d(1), \quad (7.3)$$

If $p \not\equiv 1$ (mod 10), $\Phi_5(1) = -1$, by (5.2). If $p \equiv 1$ (mod 10), $\Phi_5(1) = 5t_0$, by (9.5). Thus

$$A_5(1) = \begin{cases} 4X - 1, & \text{if } p \equiv 1 \pmod{10}, \\ -1, & \text{otherwise.} \end{cases}$$

$A_5(1)$ was given by Brewer [1]. X satisfies the equations in Theorem 8. Its sign is chosen so that $X \equiv 4$ (mod 5). That X is thus defined uniquely can be deduced from [7, Theorem 8].

10. **Conclusion.** The evaluations of Brewer sums presented here may serve as a guide to the evaluation of Brewer sums of other orders by means of cyclotomy. There may be extensions in other directions, as suggested by [3].

This paper is comprised of the principal contributions of the dissertation of the first author completed under the direction of the second, several results of the latter, and results from the dissertation, completed under Professor Albert L. Whiteman, of the third author with subsequent extensions.

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Let e = 5. Define

$$(9.5) \quad t_j = B_5(j - 2\,\mathrm{Ind}\,2, 1) - (p-1)/5; \quad t_j = \Phi_5(g^j)/5,$$

by (3.21). By setting $\lambda = 0$, $\nu = 1$ in (3.12) and comparing with (3.1) one may deduce that $\sum_{i=0}^{4} \beta_5(i, 1) = p - 2$;

$$(9.6) \quad t_0 + t_1 + t_2 + t_3 + t_4 = -1$$

is an immediate consequence.

$$R_5(1,1) = \sum_{i=0}^{4} B_5(i,1)\beta^i = \beta^{-2\,\mathrm{Ind}\,2} \sum_{j=0}^{4} [B_5(j - 2\,\mathrm{Ind}\,2, 1) - (p-1)/5]\beta^j = \beta^{-2\,\mathrm{Ind}\,2} \sum_{j=0}^{4} t_j \beta^j.$$

Since, according to (3.4), $|R_5(1,1)|^2 = p$, $|\sum_{j=0}^4 t_j \beta^j|^2 = p$, so that

$$(9.7) \quad p = \sum_{j=0}^{4} t_j^2 - A, \quad A = t_0t_1 + t_1t_2 + t_2t_3 + t_3t_4 + t_4t_0 = t_0t_2 + t_1t_3 + t_2t_4 + t_3t_0 + t_4t_1.$$

Define

$$4X = 5t_0 + 1, \quad 4U = t_1 + t_2 - t_3 - t_4,$$
$$4V = t_1 - t_2 + t_3 - t_4, \quad 4W = t_1 - t_2 - t_3 + t_4.$$

Theorem 8. *X, U, V, and W are integers satisfying $X^2 + 5U^2 + 5V^2 + 5W^2 = p$, $XW = V^2 - U^2 - UV$.*

Proof. t_0 **is odd, while t_1, t_2, t_3, and t_4 are even [15, p. 123]. From the last equation in (9.7), drop all products of two even numbers: $t_0(t_1 - t_2 - t_3 + t_4) \equiv 0$ (mod 4). Since t_0 is odd, W is an integer. Then $U = W + (t_2 - t_4)/2$ and $V = W + (t_3 - t_4)/2$ are integers. Finally by (9.6), $X = t_0 - (t_1 + t_2 + t_3 + t_4)/4 = t_0 - W + (t_2 + t_3)/2$ is an integer. Now**

$$X^2 + 5U^2 + 5V^2 + 5W^2 = [(4t_0 - t_1 - t_2 - t_3 - t_4)^2 + 5(t_1 + t_2 - t_3 - t_4)^2 + 5(t_1 - t_2 + t_3 - t_4)^2 + 5(t_1 - t_2 - t_3 + t_4)^2]/16$$
$$= \sum_{j=0}^{4} t_j^2 - \frac{1}{2}\sum_{j=0}^{3}\sum_{k=j+1}^{4} t_j t_k = p,$$

by (9.7).

To show that X, U, V, and W also satisfy $XW = V^2 - U^2 - UV$, solve for t_0, t_1, t_2, t_3, and t_4 in terms of X, U, V, W, and a constant and substitute into the last equation in (9.7).

$$\Lambda_{10}(Q) = \chi(Q)\Lambda_5(1) + \Phi_d(1) - d(p-1, 5). \quad (7.3)$$

If $p \not\equiv 1 \pmod{10}$, $\Phi_1(1) = -1$, by (5.2). If $p \equiv 1 \pmod{10}$, $\Phi_5(1) = 5t_0$, by (9.5). Thus

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